The Robertson Regulator

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1 Introduction

Bob Holmstrom has previously described the Robertson Clock at the University of Bristol, England[1]. It was built in 1925 for the then-new Wills Memorial Building in the University and was designed to keep campus time and control the chiming of Great George (which can be heard still over large areas of Bristol). David Robertson was the first professor of electrical engineering in the University, from 1911 until his death in 1941. An innovative engineer, he was also very interested in horology (mechanical clocks then keeping the world's time) and wrote a long series of articles on "The Theory of Pendulums and Escapements" in *Horological Journal* in the late 1920s and early 1930s[2]. His star student was probably Paul Dirac, who later became quite a well-known physicist. Dirac later paid tribute to Robertson's application of elegant mathematics in solving physical problems[3].

As far as we know the clock kept time from 1925 until the Great Hall suffered a bomb blast in WWII – Bristol being a prime target. The Hall was restored and presumably the clock also repaired but we don't know when its bell-ringing functions were supplanted by something more modern. When I first saw the clock in May 2016 (as a visiting staff member in the University), still in the Wills building, it had clearly not been touched for years and most of the supplementary circuitry had been removed. We started a little campaign to get it moved and restored to life as the area was due for redevelopment, and in Fall 2019 it was unveiled at its new location in the Queen's Building, home of the Engineering Faculty. More information is available from the weblinks below[4], including Robertson's own description in the Proceedings of the Institute of Shipbuilders and Engineers in Scotland[5] (a well-known horological publication…). The clock is ably described in Bob's article (which focuses on the escapement) and in Robertson's own words; in this article I want to focus on the system by which the clock attempted to synchronise itself to Greenwich time signals and adjusted its rate accordingly.

2 Robertson's regulator system

Starting from 1851[6], electrical time pulses were distributed from Greenwich throughout mainland UK using the telegraph network. By 1925 a telegraph line could be rented from the Post Office which would bring a time pulse to a premise at 10.00 am GMT every morning. This was supported by an infrastructure that took pairs out of traffic-carrying service throughout the network just before 10.00 and commoned them up to carry the pulse, then released the pairs back for traffic. This carried on until the 1950s by which time radio time signals were long established and receivers cheap and ubiquitous. Many systems were devised to synchronise clocks to these. Robertson's approach combined two key elements. The first was associated with the escapement mechanism, illustrated diagrammatically in Figure 1.

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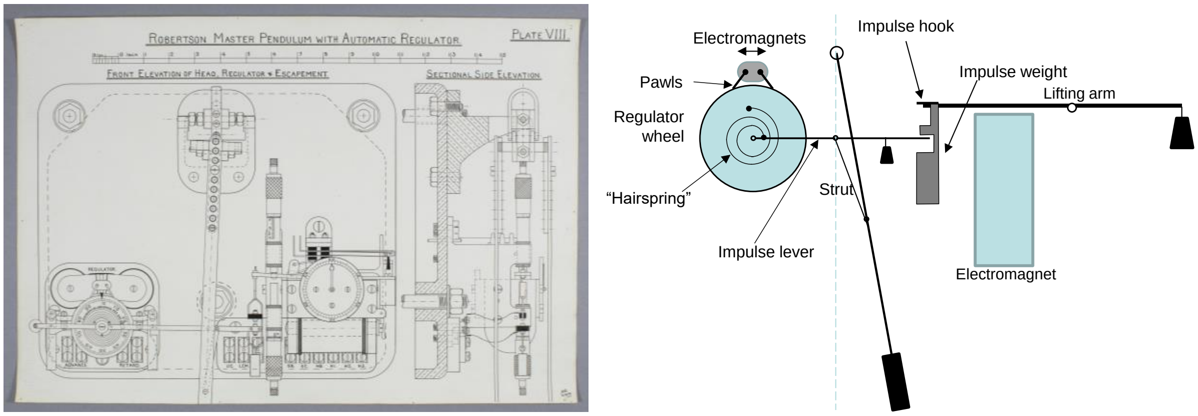


Figure 1: Left, Robertson's illustration from [3]; right, functional diagram.

The right-hand pane abstracts the functional elements of the regulator and impulsing system. The escapement and regulator both apply a force to the pendulum through an impulse lever, pivoted at the left and carrying a weight at the right so that a net compressive force is exerted on a strut which is pin-jointed at each end, on the lever and the pendulum rod. When the pendulum is vertical this force acts downwards and has no effect on its motion; but as the pendulum deviates from vertical an increasing component of the force acts to push the pendulum outwards in either direction. This force opposes the gravitational restoring force and so slows the pendulum's rate. The force on the strut can be varied by two mechanisms. One, on the right, is the impulse electromagnet which can apply extra force to the lever by lowering a weight on to its end. This impulse is applied for a short period just after bottom dead centre (BDC). The lagging impulse acts to reduce the rate by an amount that reduces with amplitude, and Robertson exploited this by operating at an amplitude that balances escapement against circular deviation, effectively at the summit of a "hill curve" (though he presents it as a valley). Thus the first-order sensitivity of rate to impulse strength is zero.

The other mechanism is a torque exerted on the lever through a spiral hairspring around the pivot axis. One end of the spring is attached to the lever, the other to a pin on a wheel, frictionally mounted so that it can rotate around (though is not coupled to) the same axis as the lever pivot. The periphery of the wheel carries 100 symmetrical teeth with a root angle of 90°, which engage with a pair of pawls mounted on the common armature of a pair of electromagnets. The pawls are arranged so that normally, when the armature is in its sprung central position, the wheel cannot rotate. By energising one magnet or the other, the pawls can step the wheel round in either direction one tooth at a time, changing the spring torque exerted on the lever, and hence also changing the pendulum's rate. A 1-tooth movement induces a rate change of 0.1 s/day. In addition, though not shown in the functional sketch, when the armature is energised in either direction, it places a small weight on the lever, to the right or left of the lever pivot depending on the magnet energised, which remains in place for a period of 5 minutes (see later). For this period the rate is accelerated or retarded to make an incremental change to the clock's time of 0.2 seconds.

It may be objected (and Frank Hope-Jones [FHJ] didn't hold back[ii]) that the strut and lever are always attached to the pendulum so it is never "free". The answer to this objection is that the forces involved are minimal, and all the friction occurs at lightly-loaded pivots and reduced by the lever effect. There are 3 pivots involved which is rather fewer than for example a grasshopper escapement. Also of course, the clock is intended to be synchronised to an external time source so absolute accuracy is not essential.

[ii] See the discussion recorded at the end of [5].

The advantages of the whole mechanism are that the escapement is highly controllable with fine independent setting of lift and drop (using micrometer thimbles!); and it gives very fine control of the rate using electrical signals applied to the regulator magnets. These are derived in the second part of Robertson's system.

In this an ingenious relay circuit compares the timing of the Greenwich pulse with a local pulse. In order to generate the local pulse, the clock is equipped with a "Greenwich Time Unit" or GTU, essentially a 24-hour counter stepped by pulses from the escapement, that generates a pulse once a day at nominally 10.00. The GTU is the mechanism shown to the right of the pendulum in the photo in [1]. Essentially it is a set of wheels stepped by electromagnet-operated ratchets, every second, minute and hour, with cams that closed contacts around 10.00 local time generating a local time pulse and various control signals. (Note the rather similar unit to the left of the pendulum which is the "Civil Time Unit" that kept GMT in winter and BST in summer (one hour ahead), and generated the signals required to chime the bell every hour between 0700 and 2100, Mondays through Saturdays.)

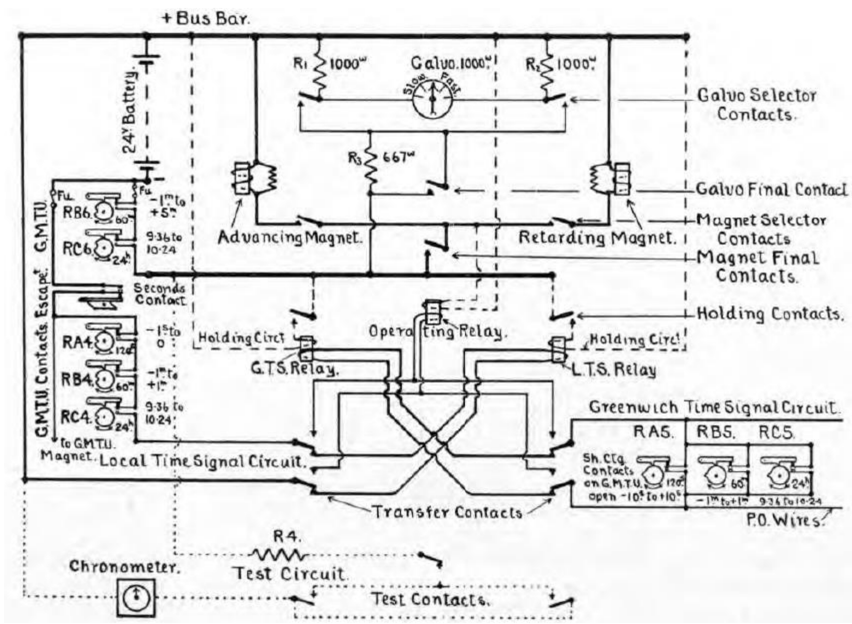


Figure 2: Time control circuit

Figure 2 is a copy of Robertson's diagram of his time comparison circuit. It has a pair of relays labelled GTS Relay and LTS Relay, GTS standing for Greenwich Time Signal and LTS standing for Local Time Signal. Each relay has two coils on its core. The circuit operates as follows.

- Shortly before 10.00 the whole circuit is "armed" ready to compare the timings[iii].
- When a time pulse arrives at its corresponding relay it energises the coil and pulls in the armature. There is a pair of contacts on each relay that would normally apply a voltage to the other coil on the core, which also magnetises the core and takes over from the first so that when the time pulse ends the relay remains energised or "latched".
- Each relay has another pair of *normally-closed* contacts in the latching circuit of the *other* relay, which can inhibit it from latching when its time pulse arrives. If one of the time pulses is earlier than the other, thus latching the corresponding relay, the other relay, though it will close

[iii] It is important to have only a narrow window in which the system is sensitive, as it will be apparent from the discussion that a spurious "Greenwich" pulse not matched by a "local" pulse, or vice-versa, will insert a time error.

momentarily when its pulse arrives, cannot latch itself. Thus, once both pulses have occurred, one relay or the other will remain latched depending on which of the GTS or LTS pulses arrived first.

- If the pulses arrive at the same moment, or within a narrow time window, each relay disables the latching of the other and neither remains energised after the pulses. This window is referred to as the “dead zone”.
- Depending which relay is latched, just after both pulses one or other of the regulator magnets is energised and retains its state for 5 minutes, after which a control signal from the GTU releases it. This has two effects: first, the regulator disc is moved one notch in an appropriate direction to increase or decrease the rate depending on whether the Greenwich or local pulse arrived first, by 0.1 seconds/day. Second, as described above, the clock is advanced or retarded by 0.2 seconds.

When I read this description in [5] I felt the hairs rise on the back of my neck! What Robertson invented and described in 1925 is perhaps the first description of what we now call a Phase-Locked Loop (PLL). Now ubiquitous, the first description in the electronics literature appeared in 1933[7]. Not until the 1970s was a “phase detector” invented[8] that works on similar lines to Robertson’s relay circuit. Until then phase detectors could only operate on signals at the same frequency, generating an error signal to phase-synchronise a local oscillator to a reference, but separate means were needed to bring the oscillator to the reference frequency. What Robertson describes is a system that later became known as a “phase-frequency detector” or PFD, sometimes referred to as a 3-state phase detector, which has the property that it can adjust the frequency of an oscillator to that of a reference and then pull it into phase lock, using one simple circuit. In electronics this can be as simple as 2 “D-type” flip-flops and a NAND gate, replacing Robertson’s relays.

3 Measured and simulated performance

Did the system work? We have little direct evidence from Robertson but an article by Ball[9] describes its performance and gives the graphs shown in Figure 3 for its timekeeping over a period of 50 days in spring 1929.

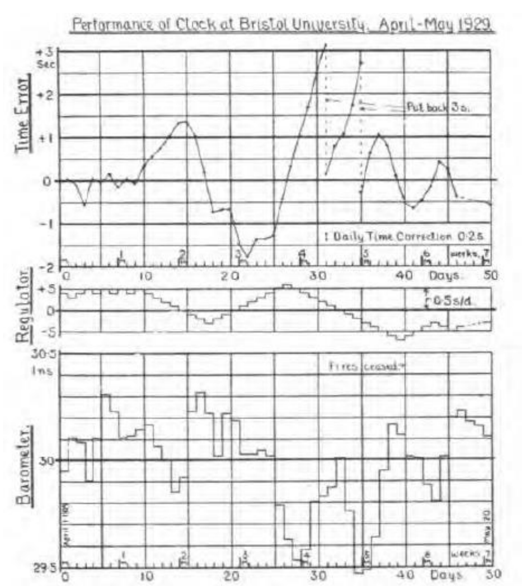


Figure 3: Performance of the regulator

The system did not seem to work very well during periods when the barometric pressure was changing rapidly. Inspection of the graphs shows that much of the time the regulator is stepping the rate up or down from day to day trying to catch up with the error that has accumulated while the pressure was quickly changing. From a modern perspective, the sampling rate of his PLL at 1 sample/day was much lower than the "bandwidth" of the pressure variations, and this was compounded by only being able to make fixed incremental rate changes. Robertson tried to improve performance using an ingenious "chronometer" that measured the actual time error and applied a proportional time shift through the regulator weights, but this proved fruitless. (The chronometer was based on a modified dc "electricity meter" and still remains in the clock case.) There is anecdotal evidence that by the 1930s the time control system had reverted[10] to "postgraduate student" placing small weights on the pendulum bob having judged the time error with reference to BBC time signals.

To explore the system's operation and barometric sensitivity I constructed a simple simulation, using Excel to model the operation from one day to the next. Here is a picture of the top of the sheet with the settings and some of the workings.

Starting rate =	0	s/day
Rate increment =	-0.1	s/day
Correction increment =	-0.2	s
Initial error =	4	s
Dead zone =	0.05	s (+/-)
Baro coeff. =	0.00E+00	

Day	Rate	Error	N	Baro rate	>Dead zone	Error sign
0	-0.500	4	5	0.00E+00	1	1
1	-0.500	2.8	6	0.00E+00	1	1
2	-0.600	1.4	7	0.00E+00	1	1
3	-0.700	-0.2	6	0.00E+00	1	-1
4	-0.600	-1.2	5	0.00E+00	1	-1
5	-0.500	-2	4	0.00E+00	1	-1
6	-0.400	-2.6	3	0.00E+00	1	-1
7	-0.300	-3	2	0.00E+00	1	-1
8	-0.200	-3.2	1	0.00E+00	1	-1

At the top one can set the starting rate (if used), the rate increment caused by one "click" of the regulator wheel, the correction increment, initial error, and a "dead zone" which is the minimum time difference to which the phase detector is sensitive. The "Day" column is essentially the day count; the "Rate" column the actual pendulum rate during that day; "Error" is the accumulated time error; and "N" is the regulator "wheel setting". A barometric rate can be added from a separate sheet, but this contribution can be turned off. The additional columns are intermediate "workings".

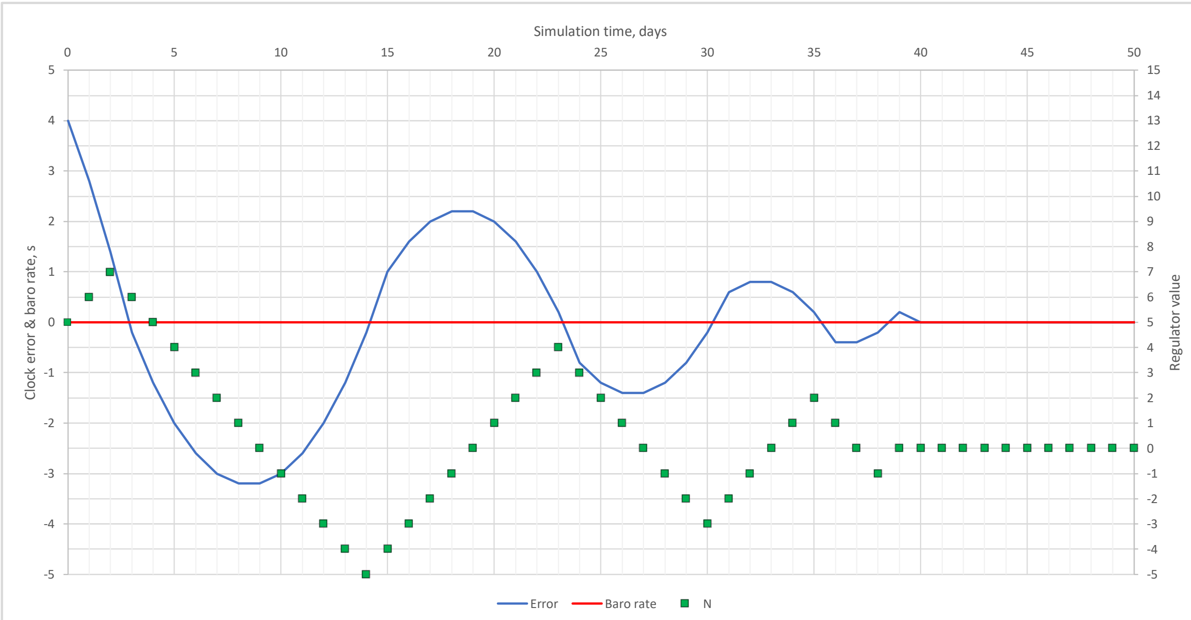


Figure 4: Simulation for initial time and rate error, no perturbation

Figure 4 shows the trajectory for the time error and regulator value for an initial time error of 4 seconds and rate error 0.5 s/day (set by the N value). One can see the system recovering from this error over 40 time samples (i.e. 40 days!), down to a point where (in this case) the time error is exactly zero – thereafter it remains at this value. (This is a numerical coincidence as the starting conditions are exact multiples of the rate increment. If a small initial rate is added the behaviour is different.) Note the "quasi sinusoid" variation of the time error – this is the integral of the rate set by N which is varying linearly as it steps up and down. The integral of a ramp is a parabola, which is the basic shape of the error curve.

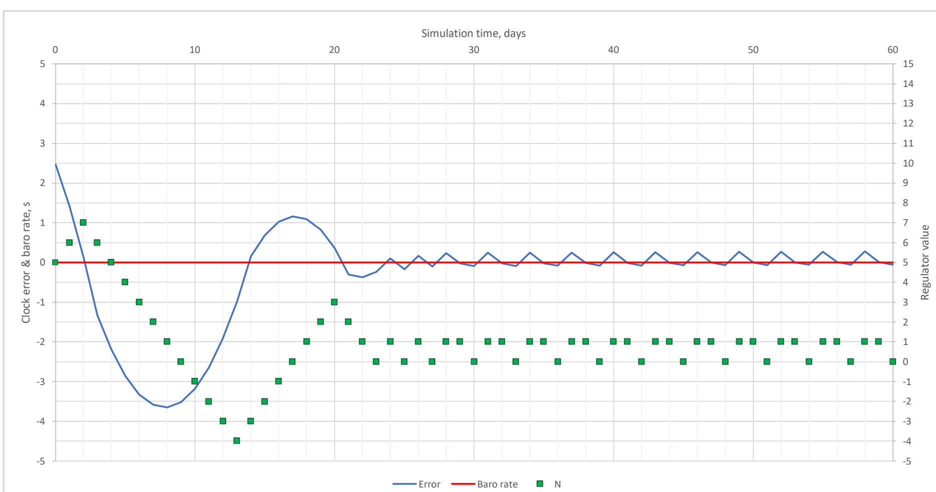


Figure 5: Settling with small initial rate error

Figure 5 shows the settling behaviour with an arbitrary initial rate of 0.1346s. Now the system enters a "limit cycle" where it has a small, quite fast error oscillation over a period of 3 days on top of a slow "sawtooth" with period 53 days, the total excursion being 0.4s. Interestingly, this magnitude does not vary much with the dead zone, remaining the same even if the latter is doubled. The magnitude also varies with the initial rate error, though not in an obviously simple way.

Since the Ball/Robertson measurements showed the clock to be apparently very sensitive to barometric pressure variations the simulation can introduce these via a separate sheet. This uses barometric data that was available from the UK's National Physical Laboratory, taken from an on-line barograph at 5 minute intervals. I have data for the period June 2015 – June 2016 which was previously used in work on the Harrison barometric compensation system. I took daily averages of this data using Excel to get 365 values and processed these to obtain the likely rate variations, assuming a lead bob and equal buoyancy and "accession to inertia" factors.

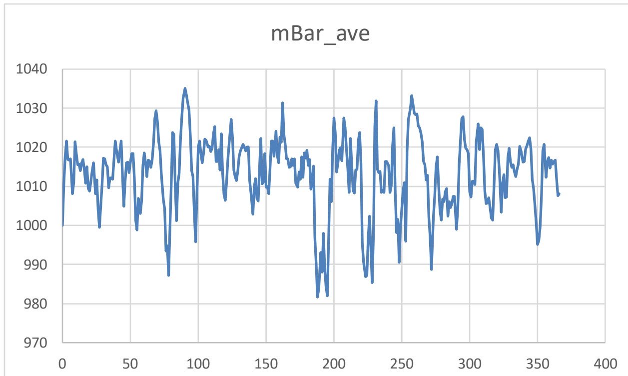


Figure 6: Daily average pressure at NPL, June '15 to June '16

Figure 6 plots the daily average pressures which show quite fast fluctuations over timescales of 1 day at several points over the year. These values are used in the simulation to derive barometric rates which are added to the simulation. Figure 7 shows the behaviour of the clock.

Figure 7: Performance with added barometric error

Despite the added "random" variation the clock still stabilises quite quickly, in fact if anything more quickly as the random changes "explore" the stable state. Thereafter the clock remains reasonably stable but interestingly it *magnifies* the barometric rate variations. The standard deviation of the barometric rate is 88ms over the year, but the clock rate SD is 253ms.

I have also tried simulating the clock using pressure data derived from Figure 3, but have not been able to fully reproduce those results. It seems like the pendulum is considerably more sensitive to barometric pressure than would be expected – this may well be because of the amplitude variations produced as the air drag changes, interacting with the lagging impulse. If the pressure increases, the drag increases, the amplitude decreases and the impulse phase lag is increased, slowing the pendulum in addition to the buoyancy changes. There will be a small reduction also in circular deviation but this could easily be outweighed by the other factors.

The performance of the regulator system is limited by two factors. First, time error is sampled only one per day; second, adjustments are limited to 0.2s absolute timing and 0.1s rate for each sample. However the barometric data shows that actual rate variations can be too fast and too large for the system to track them. It is instructive to evaluate the performance with just a simple sinusoidal pressure variation. Figure 8 shows the performance with a 30-day period variation over the same total barometric range as before. Clearly the rate of the clock (as seen by the green marker dots that delineate the regulator setting) is tracking the rate reasonably well but there is a random-ish error variation which amounts to 0.8s peak-to-peak.

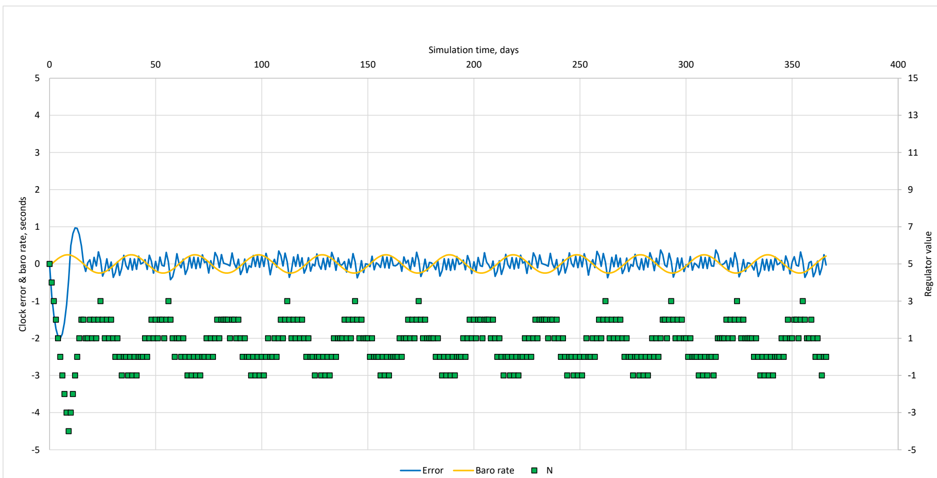


Figure 8 Performance with slow pressure variation

Overall it can be concluded that the regulator system is reasonably effective in getting the clock to time even with quite large initial errors in rate and phase; but is not so good at accurately tracking random fluctuations caused by barometric pressure changes. The errors are seldom likely to exceed one second except for very short periods, which would have been more than adequate for its intended function[11].

4 Restoration and further work

In 2017 the clock was carefully dismantled under the direction of Johan Ten Hoeve of "The Clockworks"[12] and taken to their workshop in London where Johan and James Harris dismantled, cleaned and reassembled all the mechanisms. We decided to focus on the pendulum, the escapement and regulator mechanism, and the dial. The clock was installed in its new location and inaugurated in October 2019 where it continues to keep time. One reason for not trying to restore the GTU and CTU is that these are quite noisy in operation being driven by large solenoids – the wall on which the clock is mounted continues to the next floor to a lecture theatre and we thought the continuous clicking would be unacceptable. Since neither the regulator nor the bell are being operated from the clock there is no need for the GTU/CTU functions.

Some of the functionality of the escapement and regulator have been lost. The escapement includes a ratchet count wheel stepped by the drive magnet, and wear in its pawl makes this slightly unreliable and prone to stalling. Since this operates a pair of contacts twice a minute to drive slave clocks which actually display the time, this is a bit of a problem! The decision was therefore made to add some modern assistance to the clock in the form of an Arduino which currently just acts as a counter to drive the slave dial. Further software functions, including driving the regulator, may be added in future.

We kept the contacts which sense when the pendulum reaches its centre position to trigger the impulse. These are operated by the regulator lever, closing just before it reaches its highest position. However the clock as it is today is not as it was drawn and originally built (possibly because of hasty repairs to bomb damage), and the contacts are not very satisfactory. They are not well aligned and prone to get out of position. Though we have protected against arc erosion by using just a resistor to provide wetting current rather than having the contacts switch the magnet current directly, in the long term it would be desirable for long life and low maintenance to replace the contacts with an opto-interrupter sensing the pendulum finial, with suitable firmware in the Arduino to handle the signal. This would also permit easy monitoring of the pendulum timekeeping. Though changes like this are not in the spirit of "conservation" we believe that Robertson would have approved.

On the regulator, we know that Robertson had problems with it and had to add some damping to its armature in the form of a leather friction pad. In addition we have found that one of the magnet coils has gone open-circuit – repairing this would require completely dismantling the unit to rewind it. One of the reasons for restoring the clock was to provide a platform for student projects and an obvious subject would be to look at the regulator, which is frustrated by it being broken! One of my project students, Mr. Minyu Zhang, undertook a project to investigate the regulation.

Since the original "ratchet stepper motor" is unusable an alternative way to drive the regulator wheel is needed. In Robertson's time electromagnetically driven pawls driving ratchets were common and he used these in the escapement for the seconds counter, to drive the CTU and GTU, as well as the regulator. Had modern stepping motors been around I'm sure he would have enthusiastically adopted them instead and this is what we chose to do. A NEMA 8 stepper motor carries a cleverly designed pinion, laser cut from acrylic sheet with 10 semi-circular teeth, which engage with the right-angled tooth spaces on the regulator wheel. This gives a 10:1 reduction between stepper and the wheel so the motor has more than adequate torque even using a low drive current. The stepper is carried by a platform supported by a structure based on the traditional Chinese Sunmao principles[13] which can be mounted to the clock without any need to alter its structure (such as drilling holes), or indeed do anything to the regulator other than ensure that the pawls are disengaged from the wheel. The photos below show the pinion (a push fit on the motor shaft) and the motor on its mounting installed on the clock.

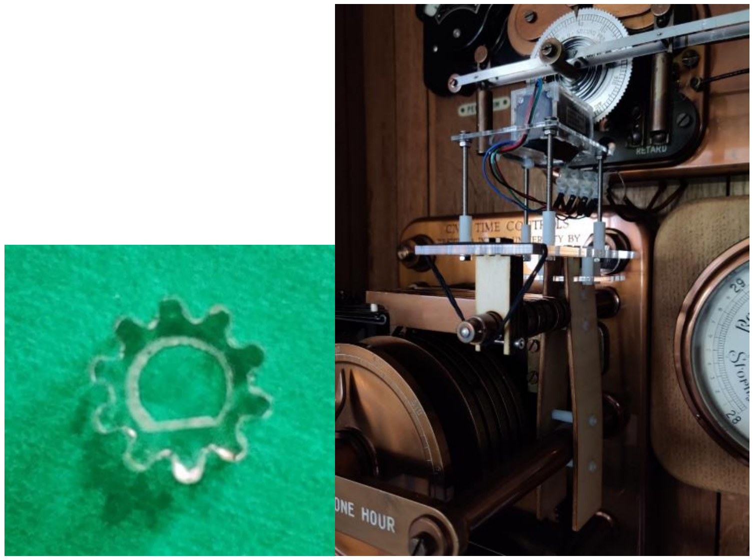


Figure 9: (a) Stepper pinion and (b) motor mount

Unfortunately the practical phase of Mr. Zhang's project was interrupted by the COVID-19 pandemic though the mechanical principles were proven. His conceptual, design and construction work though is outstanding. Next steps with the clock will be to replace the escapement contacts with the opto device and continue work on the regulation system, with the objective of determining if Robertson's system, with more frequent sampling, could have tracked barometric variations more accurately.

5 Robertson's legacy

Robertson's writings in HJ are probably one of the most complete presentations of pendulum and escapement theory, and though somewhat inaccessible they are nevertheless sometimes quoted in these pages. In his time Robertson also made major contributions to electrical engineering and engineering education. It is well worth reading the interviews with Dirac in [3] – he speaks a lot about Robertson and how Robertson's use of mathematics and approximations influenced the thinking of a great physicist.

Robertson was a pioneer in disabled rights, having been paralysed from the waist down by illness in around 1911, around the time he joined the Bristol faculty, so he carried out all his professorial duties from a wheelchair from 1911 to his death in 1941, and this at a time when few buildings (certainly not those of the University at that stage) were well adapted. (Bristol itself, especially the University location, is far from being flat!).

Unlike the Synchronome, Robertson's clock is in a sense a horological dead-end because of its appearance at the twilight of mechanical horology. It is clearly a prototype – who would design a production clock with a couple of Starrett micrometer heads for adjustment? And reading Robertson's account in [5] we can hear the frustration of an engineer chasing down highly intermittent "bugs" in a complex one-off system. By 1926 probably thousands of Synchronomes were giving good service, but no doubt in their early stages they had many teething troubles too.

Robertson's clock is an interesting example of how an engineer with a deep knowledge of horology could rethink clock design. When it appeared in 1926, the Synchronome had dominated the market for institutional clocks for some time, and similar clocks were emerging for use in telephone exchanges. One must also remember that in the 1930s Louis Essen was working on quartz clocks – the Essen Ring clock was launched in 1938 and soon took over from clocks such as the Shortt-Synchronome as the national time standard. So the Robertson clock came perhaps too late to have much influence on precision timekeeping. Designed by an engineer, we have a complete set of engineering drawings for the clock, which was manufactured by a Bristol company, Brecknell, Munro and Rogers, Ltd "mechanical, electrical, and tramway engineers".

An interesting thought: the precision of the Shortt-Synchronome is limited since it doesn't actually adjust the frequency of the slave pendulum, but just "nudges" its phase if it is slightly slow compared to the master reference every 30 seconds. Experience shows that setting up such a system is very tricky as the slave has to be brought very close to time but with a very slightly longer period. But using Robertson's regulator system, the slave pendulum frequency could automatically have been adjusted and kept precisely in synch based on pulses every 30s from the Shortt master pendulum. The PLL comparison frequency would be increased by a factor of 2880 and it could probably have coped well with rapid rate changes caused by barometric variations. Alas we will probably never have the opportunity to investigate the hybrid "Shortt – Robertson" clock!

Robertson invented the essentials of the phase-locked loop and 3-state phase detector, though he didn't perhaps fully understand the ramifications. His work was in the public domain in 1926, but the first recognised patent for the phase-locked loop wasn't filed in France until 1931 – an alert and well-informed patent examiner might have cited Robertson as prior art. Likewise, the first descriptions and patents on sequential phase detectors appeared in the early 1970s and Robertson's relay logic circuit would have been prior art. The performance of the system as we have seen is limited because of the very low sampling rate and the small, fixed control increment on each correction. Robertson was working at a time when the fundamentals of feedback servo systems were little known, and sampled-data and incremental systems were far into the future. He had no way to record long sequences of numerical data, for example barometer readings, for detailed analysis to understand implications for control strategy. And while a simple simulation in Excel suffices to demonstrate the system performance in considerable detail, such methods were completely unknown in his time.

Robertson's mechanism for adjusting a pendulum is very interesting. It provides a way in which its rate can be automatically adjusted in fine increments, for example by a stepper motor. Robertson just wanted a way his clock could be kept in sync with Greenwich from day to day without human intervention – nowadays if we want accurate local time, or UTC, we can just use our phones or glance at the bottom right of our PC screen. There would be little point in phase-locking a pendulum to an external timing source. But we could envisage a clock with a pendulum kept in motion at constant amplitude by an electromagnet system with optical sensing, with no compensation for temperature, pressure, or humidity. Then a microprocessor would use environment data from a cheap integrated sensor such as the Bosch BME280[14] to adjust the pendulum rate using predetermined regression coefficients through a Robertson-style mechanism. The Appendix analyses the regulator to show how the rate depends on the geometry and the applied force.

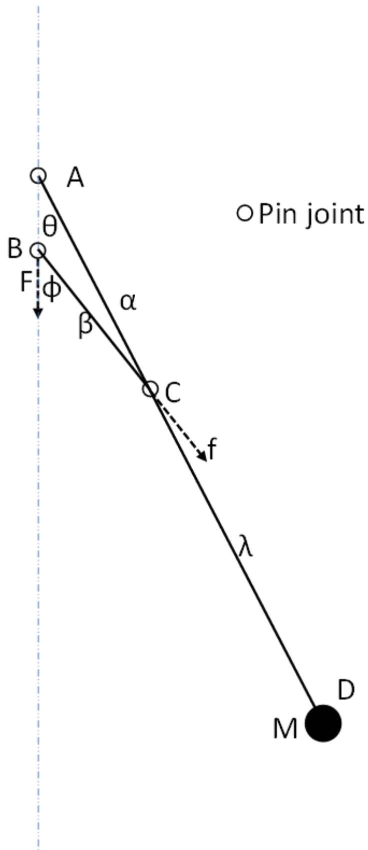
Acknowledgements
I'd like to thank everyone in the University who has helped to rescue the Robertson Clock, especially Alan Stealey and the staff of the Engineering Faculty. Johan Ten Hoeve and James Harris did an excellent job bringing the clock back to life. I'd also like to thank Mr. Minyu Zhang for his work on the regulator and permission to use his photographs. Dr. Andrew Millington kindly set me right on the analysis of the linkage presented in the Appendix and made other useful comments on an early draft.

Appendix: analysis of the Robertson Regulator

Consider the pendulum and linkage arrangement shown in the figure (left).

The rod of total length λ is pivoted at A and has a bob of mass M at D. A strut BC is pin jointed to the rod at C; while end B is also pin jointed and constrained to move only on the vertical axis through A. A vertical force F acts at the pin joint B. Point C is at a length α on the rod below the pivot A and the link length is β. It is evident that when the pendulum displacement is θ, point B has moved down from its maximum height α-β, and the angle ϕ between the link and the vertical will be greater than θ.

Assuming that motion is slow enough for a quasi-static solution, the downward force F must be counterbalanced by a compressive force *f* in the strut, such that



$$f\cos\phi = F\,; => f = {}^{F}\!/_{\cos\phi}$$

This force acts on the rod at C as shown by the vector ***f*** and must be counterbalanced by components along and orthogonal to the rod. The angle between ***f*** and the rod is equal to the angle BCA and given by:

$$ang(BCA) = \pi - (\pi - \phi) - \theta = \phi - \theta$$

To find angle ϕ note that the perpendicular distance of C from the axis is given by:

$$\beta\sin\phi = \alpha\sin\theta$$

So

$$\sin\phi = \frac{\alpha}{\beta}\sin\theta => \phi = \sin^{-1}\left(\frac{\alpha}{\beta}\sin\theta\right)$$

The component of force orthogonal to the rod at C is then:

$$\frac{F}{\cos\left(\sin^{-1}\frac{\alpha}{\beta}\sin\theta\right)}.\sin\left(\sin^{-1}\left(\frac{\alpha}{\beta}\sin\theta\right) - \theta\right)$$

According to Wolfram Alpha:

$$\frac{\sin(\sin^{-1}(a.\sin(x)) - x)}{\cos(\sin^{-1}(a.\sin(x)))} = -\frac{\sin(x - \sin^{-1}(a.\sin(x)))}{\sqrt{1-(a.\sin(x))^2}}$$

And this can be expanded around x = 0 as:

$$\frac{a.0}{\sqrt{1-(a.0)^2}} + \left(\frac{a.1}{(1-(a.0)^2)^{3/2}} - 1\right)x - \frac{(a.0\,((a.0)^4 - 2\,(a.0)^2 - 3\,(a.1)^2 + 1))\,x^2}{2\,(1-(a.0)^2)^{5/2}} -$$
$$\frac{1}{6\,(1-(a.0)^2)^{7/2}}\left(\sqrt{1-(a.0)^2}\,(a.0)^6 + \left(4\,a.1 - 3\sqrt{1-(a.0)^2}\right)(a.0)^4 + \right.$$
$$\left.\left(-12\,(a.1)^3 - 8\,a.1 + 3\sqrt{1-(a.0)^2}\right)(a.0)^2 - 3\,(a.1)^3 + 4\,a.1 - \sqrt{1-(a.0)^2}\right)x^3 +$$

for terms up to the 3rd degree in x. Simplifying, the coefficients of the zero'th and second-degree terms are identically zero and up to degree 3 in x the expression can be written

$$(a-1)x - {}^{1}/_{6}\{4a - 3a^3 - 1\}x^3 + \varepsilon(x^5)$$

For small x, the cubic and higher degree terms can be ignored.

For the geometry shown $a = {}^{\alpha}/_{\beta}$ so the torque on the pendulum for a small angle is:

$$F\alpha\left(\frac{\alpha}{\beta} - 1\right)\theta$$

This acts in opposition to the restoring torque of the bob so the total torque can be written:

$$\left\{Mg\lambda - F\alpha\left(\frac{\alpha}{\beta} - 1\right)\right\}\theta = M\lambda\left\{g - \frac{F\alpha}{M\lambda}\left(\frac{\alpha}{\beta} - 1\right)\right\}\theta$$

Thus the effective value of *g* is reduced by an amount:

$$\frac{F\alpha}{M\lambda}\left(\frac{\alpha}{\beta} - 1\right)$$

The fractional reduction in *g* is then:

$$\frac{F\alpha}{M\lambda g}\left(\frac{\alpha}{\beta} - 1\right)$$

and the fractional increase of period:

$$\boldsymbol{\frac{F\alpha}{2M\lambda g}\left(\frac{\alpha}{\beta} - 1\right)}$$

If the force was applied by a small weight *m* the increase in period would be:

$$\frac{m\alpha}{2M\lambda}\left(\frac{\alpha}{\beta} - 1\right) = \boldsymbol{\frac{1}{2}\frac{m}{M}\frac{\alpha}{\lambda}\left\{\frac{\alpha}{\beta} - 1\right\}}$$

If the link length β was equal to *α* then the applied force will produce no change in rate, as expected.

Robertson's clock has dimensions:

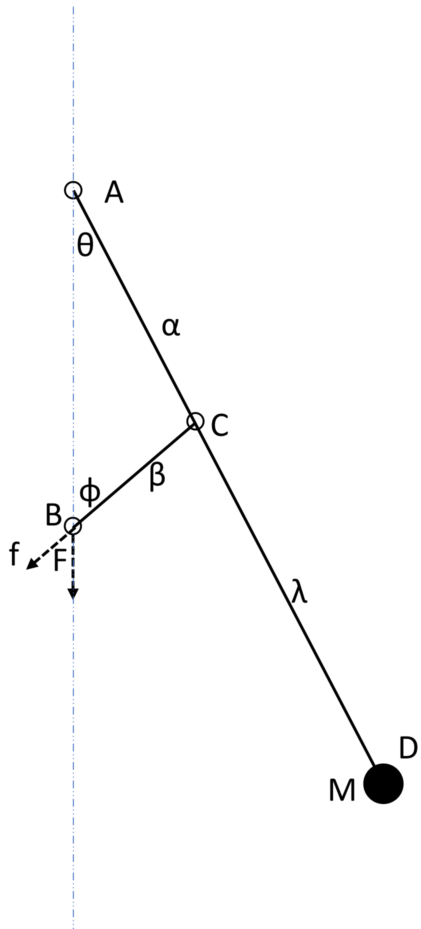
α = 10 inches; β = 2.5 inches; λ = 39.875 inches; and bob weight is about 7.5 kg. Thus:

$$\frac{\alpha}{\lambda} \approx 0.25; \frac{\alpha}{\beta} = 4$$

The effect of a 1 gram weight applied at B would then be to produce a rate change of:

$$-\frac{1}{2}.\frac{1}{7500}.0.25\{4-1\} = -\frac{1}{8}.\frac{1}{1500} = -83.33 \times 10^{-6} \equiv -7.2 \text{ seconds per day.}$$

Robertson states in [5] that one click on the regulator disc (1/100 revolution) changes the rate by 0.1 s/day, so the change in force "per click" must be $\frac{.1}{7.2} \approx 14\text{mg}$.



An alternative configuration of the regulator is possible, where the force F is exerted below the strut so the latter is in tension, which pulls the pendulum towards BDC rather than pushing it away, causing a reduction in rate rather than an increase, as shown to the left using the identical symbols. It is easy to show that exactly the same equations apply to this except that the force, and rate change, is of opposite sign. This configuration is interesting as a strut in tension may be mechanically more stable – the Robertson clock has a small issue in that the adjustment of the strut bearings for acceptable fore-and-aft shake but minimal friction is quite critical, since the strut will tend to "topple" slightly if there is any play in the lower bearings. It is even possible that the strut could be replaced with a thin cord or wire in tension, eliminating the bearings completely.

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[11] After submitting this manuscript Andrew Millington and I have done some further simulations that show how the much larger excursions may arise – this will be reported in a later article.

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